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1996 J. Phys. A: Math. Gen. 29 2435

(http://iopscience.iop.org/0305-4470/29/10/021)

J. Phys. A: Math. Gen. **29** (1996) 2435–2438. Printed in the UK

Dynamical critical exponent of a double chain of spins

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Received 6 November 1996

Abstract. We study the dynamical critical behaviour of a double chain of spins where we take into account short-range four-spin interactions. This system is exactly soluble only in the thermodynamical equilibrium. We use the initial response rate of the order parameter to establish a lower bound to the dynamical critical exponent z. We show that, for the one- and two-spin-flip Glauber transition rates, the exponent z depends on the microscopic details of the Hamiltonian, that is, the dynamical critical exponent appears non-universal. This type of non-universal behaviour has already been seen in the one-dimensional Ising model with non-uniform exchange interactions.

In this work we study the dynamical critical behaviour of a double chain of spins. Our main interest here is to obtain a lower bound to the dynamical critical exponent of this model. In figure 1, we exhibit the model we consider, along with the exchange interactions. The Hamiltonian of this model can be written as

$$H = -J_1 \sum_{i=1}^{N} (\sigma_{i,1}\sigma_{i,2}\sigma_{i+1,1}\sigma_{i+1,2}) - J_2 \sum_{i=1}^{N} (\sigma_{i,1}\sigma_{i+1,1} + \sigma_{i,2}\sigma_{i+1,2}) - \frac{J_3}{2} \sum_{i=1}^{N} (\sigma_{i,1}\sigma_{i,2} + \sigma_{i+1,1}\sigma_{i+1,2}) \tag{1}$$

where J_1 is the short-range four-spin exchange interaction, J_2 is the exchange coupling between nearest-neighbour spins in each chain, and J_3 is the interchain exchange coupling between nearest-neighbour spins. We assume that $\sigma_{i,\alpha} = \pm 1$, where $\alpha = 1, 2$ labels the upper and the lower chain, respectively. This Hamiltonian naturally appears when we integrate the elastic degrees of freedom of a compressible Ising double chain in the pressure ensemble [1]. In equilibrium, the model given by the Hamiltonian, equation (1), is exactly soluble. The canonical partition function of this model can be easily determined through the transfer matrix method.

To study the dynamical properties of this model, we follow the method outlined by Glauber [2], in which the variables of spins are stochastic functions of time. The system, in contact with a thermal bath at temperature T, evolves in time from a non-equilibrium state to equilibrium through the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\sigma_{1,1}, \ldots, \sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{N,2}, t) = -\sum_i w_i(\sigma_{i,1}, \sigma_{i,2}) P(\sigma_{1,1}, \ldots, \sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{N,2}, t) + \sum_i w_i(-\sigma_{i,1}, -\sigma_{i,2}) P(\sigma_{1,1}, \ldots, -\sigma_{i,1}, -\sigma_{i,2}, \ldots, \sigma_{N,2}, t) \tag{2}$$

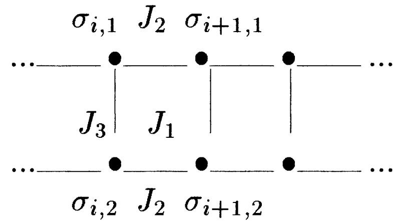

Figure 1. A schematic representation of the double chain of spins. J_1 is the short-range four-spin interaction, J_2 is the longitudinal exchange coupling and J_3 is the exchange coupling for each rod of the double chain.

where $P(\sigma_{1,1},\ldots,\sigma_{i,1},\sigma_{i,2},\ldots,\sigma_{N,2},t)$ is the probability of finding the system in the state $(\sigma_{1,1},\ldots,\sigma_{i,1},\sigma_{i,2},\ldots,\sigma_{N,2})$ at time t, and $w_i(\sigma_{i,1},\sigma_{i,2})$ gives the probability, per unit time, for the transition from the state $(\sigma_{i,1},\sigma_{i,2})$ to $(-\sigma_{i,1},-\sigma_{i,2})$. We use the detailed balance condition (DBC) to find an explicit expression for $w_i(\sigma_{i,1},\sigma_{i,2})$. Then we can write

$$\frac{w_i(\sigma_{i,1},\sigma_{i,2})}{w_i(-\sigma_{i,1},-\sigma_{i,2})}=\frac{P_{\text{eq}}(-\sigma_{i,1},-\sigma_{i,2})}{P_{\text{eq}}(\sigma_{i,1},\sigma_{i,2})} \tag{3}$$

where

$$P_{\text{eq}} \propto \mathrm{e}^{-\beta H} \tag{4}$$

is the equilibrium probability and $\beta = 1/k_{\text{B}}T$ (k_{B} is the Boltzmann constant). After some algebraic manipulations we find the following expression for the transition probability:

$$\begin{aligned} w_i(\sigma_{i,1},\sigma_{i,2}) = {} & \tfrac{1}{2}\alpha[1-\tfrac{1}{2}\gamma_2\sigma_{i,1}(\sigma_{i+1,1}+\sigma_{i-1,1})-\tfrac{1}{2}\gamma_2\sigma_{i,2}(\sigma_{i+1,2}+\sigma_{i-1,2}) \\ & +\tfrac{1}{4}\gamma_2^2\sigma_{i,1}\sigma_{i,2}(\sigma_{i+1,1}\sigma_{i+1,2}+\sigma_{i+1,1}\sigma_{i-1,2}+\sigma_{i-1,1}\sigma_{i+1,2}+\sigma_{i-1,1}\sigma_{i-1,2})] \end{aligned} \tag{5}$$

where $\gamma_2 = \tanh 2K_2 = \tanh(2J_2/K_{\text{B}}T)$ and α is a constant. It is important to note that our transition probability is slightly different from the one proposed by Glauber [2], because here we flip two spins of a given rod at the same time.

Unfortunately, with the expression of $w_i(\sigma_{i,1},\sigma_{i,2})$, it is not possible to find exact expressions for the evolution of the magnetization and of the high-order correlation functions from the master equation. This occurs because on the right-hand side of the master equation there appear high-order correlation functions. The dynamical scaling hypothesis [3] asserts that, near the critical point,

$$\tau_{\boldsymbol{q}} = \xi^z F(\boldsymbol{q}\xi) \tag{6}$$

where $\tau_{\boldsymbol{q}}$ is the relaxation time of the order parameter, $\boldsymbol{q}$ is the critical wave vector, ξ is the static correlation length, and z is the so-called dynamical critical exponent. The function $F(\boldsymbol{q}\xi)$ is an analitical function of its argument. Based on this scaling relation, and employing the method of initial response rate of the order parameter [4], we can derive a rigorous lower bound to the dynamical critical exponent z. After a straightforward calculation we obtain the following lower bound for the relaxation time of the order parameter:

$$\tau_{\boldsymbol{q}} \geqslant \frac{k_{\text{B}}T\chi_{\boldsymbol{q}}}{(4/N)\sum_{j=1}^{N}\langle(1+\sigma_{j,1}\sigma_{j,2})w_j(\sigma_{j,1},\sigma_{j,2})\rangle_{\text{eq}}} \tag{7}$$

where

$$\chi_{\boldsymbol{q}} \equiv \frac{1}{N}\sum_{j,k}\langle(\sigma_{j,1}+\sigma_{j,2})(\sigma_{k,1}+\sigma_{k,2})\rangle\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}_k-\boldsymbol{r}_j)} \tag{8}$$

is the static magnetic susceptibility corresponding to the wave vector $\boldsymbol{q}$.

Therefore, it is possible to find a lower bound to the dynamical critical exponent, defined previously through equation (6). In order to derive this lower bound, it is only necessary to evaluate the right-hand side of the inequality (7) at the thermodynamical equilibrium.

Using expression (5) for the transition probability, the average value in the denominator of inequality (7) can be written as

$$\begin{aligned}\langle(1+\sigma_{i,1}\sigma_{i,2})w_i(\sigma_{i,1},\sigma_{i,2})\rangle_{\mathrm{eq}} = \tfrac{1}{2}\alpha\{1 &+ \langle\sigma_{i,1}\sigma_{i,2}\rangle \\ &-\tfrac{1}{2}\gamma_2[\langle\sigma_{i,1}\sigma_{i-1,1}\rangle+\langle\sigma_{i,1}\sigma_{i+1,1}\rangle+\langle\sigma_{i,2}\sigma_{i-1,2}\rangle+\langle\sigma_{i,2}\sigma_{i+1,2}\rangle+\langle\sigma_{i,2}\sigma_{i-1,1}\rangle \\ &+\langle\sigma_{i,2}\sigma_{i+1,1}\rangle+\langle\sigma_{i,1}\sigma_{i-1,2}\rangle+\langle\sigma_{i,1}\sigma_{i+1,2}\rangle] \\ &+\tfrac{1}{4}\gamma_2^2[\langle\sigma_{i+1,1}\sigma_{i+1,2}\rangle+\langle\sigma_{i+1,1}\sigma_{i-1,2}\rangle+\langle\sigma_{i-1,1}\sigma_{i+1,2}\rangle+\langle\sigma_{i-1,1}\sigma_{i-1,2}\rangle \\ &+\langle\sigma_{i,1}\sigma_{i,2}\sigma_{i+1,1}\sigma_{i+1,2}\rangle+\langle\sigma_{i,1}\sigma_{i,2}\sigma_{i+1,1}\sigma_{i-1,2}\rangle+\langle\sigma_{i,1}\sigma_{i,2}\sigma_{i-1,1}\sigma_{i+1,2}\rangle \\ &+\langle\sigma_{i,1}\sigma_{i,2}\sigma_{i-1,1}\sigma_{i-1,2}\rangle]\}_{\mathrm{eq}}. \end{aligned} \tag{9}$$

The correlation functions in equation (9) are easily evaluated, as well as the magnetic susceptibility and the correlation length, through the transfer matrix method [1]. The critical wave vector for the double chain of ferromagnetic spins correspond to the uniform situation, that is to say $\boldsymbol{q}=0$, and the magnetic susceptibility $\chi_{q=0}$ diverges at $T\to 0$. Therefore, we can find the asymptotic average values in equation (9) in the limit $T\to 0$. After some algebraic manipulations we can show that

$$\langle(1+\sigma_{i,1}\sigma_{i,2})w_i(\sigma_{i,1},\sigma_{i,2})\rangle_{\mathrm{eq}} \approx 2\alpha\mathrm{e}^{-4K_2}. \tag{10}$$

When the values of the correlation length and of the magnetic susceptibility [1] are used, along with equations (6) and (7), the inequality (7) gives $z\geqslant 2$, as the lower bound to the dynamical critical exponent associated with a transition rate that takes into account the simultaneous flipping of two spins belonging to the same rod. As to be expected, this result is equal to the one obtained for the one-dimensional Ising model with uniform exchange couplings between nearest neighbours. This is reasonable because the flipping of a single vertical rod of spins of the double chain is equivalent to flipping a single spin in the linear chain.

On the other hand, we obtain a different lower bound if we choose another type of transition rate for the double chain. For instance, the simultaneous flipping of two neighbouring spins on the same chain leads to the following transition probability:

$$\begin{aligned}w_i(\sigma_{i,1},\sigma_{i+1,1}) = \tfrac{1}{2}\alpha[1 &- \tfrac{1}{2}\gamma_1(\sigma_{i-1,1}\sigma_{i-1,2}\sigma_{i,1}\sigma_{i,2}+\sigma_{i+1,1}\sigma_{i+1,2}\sigma_{i+2,1}\sigma_{i+2,2})] \\ &\times[1-\tfrac{1}{2}\gamma_2(\sigma_{i-1,1}\sigma_{i,1}+\sigma_{i+1,1}\sigma_{i+2,1})][1-\tfrac{1}{2}\gamma_3(\sigma_{i,1}\sigma_{i,2}+\sigma_{i+1,1}\sigma_{i+1,2})]\end{aligned} \tag{11}$$

where $\gamma_i=\tanh 2K_i$, with $i=1,2,3$ and $K_i=J_i/K_{\mathrm{B}}T$. The use of initial response rate of order parameter provides an inequality similar to that of equation (7). However, the asymptotic average values in the low temperature limit give

$$\langle(1+\sigma_{j,1}\sigma_{j,2})w_j(\sigma_{j,1},\sigma_{j+1,1})\rangle_{\mathrm{eq}} \approx 16\alpha\mathrm{e}^{-4K_1-4K_2-4K_3}. \tag{12}$$

Then, we obtain the following inequality to the dynamical critical exponent z:

$$z \geqslant 2+\frac{J_1}{J_2}+\frac{J_3}{J_2}.$$

Surprisingly, this result shows a dependence on the microscopic details of the Hamiltonian. Only if J_1 and J_3 vanish, do we obtain the expected value $z\geqslant 2$. This type of non-universal behaviour of the dynamical critical exponent has already been seen for the one-dimensional Ising model with non-uniform exchange couplings [5–7]. If, instead of flipping two spins at the same time, we could flip only a single spin. The result we obtain, after a straigthforward calculation, is given by

$$z \geqslant 2 + \frac{J_1}{J_2} + \frac{J_3}{2J_2}$$

where the dependence on the details of the Hamiltonian is again exhibited. Haake and Thol [8] also have shown that for one-dimensional Ising models the critical dynamical exponent z can depend on the details of the spin-flip transition rate, but Southern and Achiam [9] have studied the reasons for the violation of the dynamical scaling in these one-dimensional models. These one-dimensional Ising systems have at the critical point a zero temperature. For an inhomogeneous Ising chain there appears metastable states that are frozen against single spin-flips at very low temperatures. In fact, these metastable states contribute with infinite relaxation times, and are due to short-range interactions. However, if we include these non-critical contributions the dynamical critical exponent is overestimated. Southern and Achiam [9] have shown that the standard dynamics is recovered for the one-dimensional Ising model with alternating bonds if the two-spin-flip dynamics is considered. In the case of our double chain of Ising spins the metastable states are frozen even for two-spin flips. This is the reason for the large value found for the lower bound of the dynamical critical exponent. As pointed out by Southern and Achiam these frozen non-critical metastable states can be excited only by considering multiple-spin flips. Perhaps, for the system we are considering, it would be necessary to consider more than four-spin flips simultaneously in order to recover the standard dynamics.

To summarize, the method of initial response rate of the order parameter was applied to a double chain of spins in order to derive a rigorous lower bound to its dynamical critical exponent. We have shown that if we flip only a single spin, or two neighbouring spins inside the same chain, the dynamical critical exponent z appears to be non-universal. The dependence of z on the parameters of the Hamiltonian also appears in one-dimensional Ising models with random exchange couplings. On the other hand, when we flip two spins on the same vertical rod of the double chain, the exponent z appears universal. This can be understood as the two spins on each rod were collapsed onto a single spin, and the double chain changed to a linear chain. A universal behaviour for the dynamical critical exponent of the double chain of spins, with short-range four-spin interactions, can be obtained if multiple-spin flips are considered as pointed out by Southern and Achiam.

Acknowledgments

We wish to acknowledge the Brazilian agencies CNPq and CAPES for their financial support of this work.

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